Math 210C Lecture 20 Notes

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1 Ext Functors, Extensions, and Group Cohomology

1.1 Exactness of the localization functor

Proposition 1.1. *If R is a ring and $S \subseteq R$ is multiplicatively closed, then localization by S is an exact functor R-Mod $\to S^{-1}R$-Mod.*

Corollary 1.1. *$S^{-1}R$ is a flat R-module.*

Proof. $S^{-1}A \cong S^{-1}R \otimes_R A$. $\square$

1.2 Ext functors

Definition 1.1. let R, S be rings, and let A be an R-S-bimodule. The **i-th Ext functor** $\mathrm{Ext}^i : R\text{-Mod} \to S\text{-Mod}$ is the i-th right derived functor of $h_A = \mathrm{Hom}_R(A, \cdot)$.

Example 1.1. Let $R = \mathbb{Z}$ and $B = \mathbb{Z}/n\mathbb{Z}$. Then we have the injective resolition

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\ n\ } \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

To find $\mathrm{Ext}^i_{\mathbb{Z}} *A, \mathbb{Z}/n\mathbb{Z})$, we have

$$\underbrace{\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})}_{=:A^\vee} \xrightarrow{n} \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}, \mathbb{Z})$$

Then

$$\mathrm{Ext}^i_{\mathbb{Z}}(A, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} A^\vee[n] & i = 0 \\ A^\vee/nA^\vee & i = 1 \\ 0 & i \geq 2. \end{cases}$$

The functor $h^B = \mathrm{Hom}_R(\cdot, B) : (R\text{-Mod})^{\mathrm{op}} \to \mathrm{Ab}$ is right exact.

Theorem 1.1. *Let A be an R-module. Then $\mathrm{Ext}^i_R(A, B) \cong H^i(\mathrm{Hom}_R(P_\cdot, B))$, where $P_\cdot \to A$ is a projective resolution by R-modules.*

1.3 Extensions of modules

Definition 1.2. An **extension** of an R-module B by an R-module A is an exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

of R-modules.

Remark 1.1. Ext measures extensions.

Definition 1.3. Two extensions E, E' of B by A are **equivalent** if there is an isomorphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

This is an equivalence relation. We denote the equivalence classes as $\mathcal{E}(B, A)$.

Example 1.2. Let $A = B = \mathbb{Z}/p\mathbb{Z}$ in Ab. Then we have the p (inequivalent) extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{pi} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where $1 \leq i \leq p-1$. So $\mathcal{E}(A, B)| = p$, and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.

How do we go back and forth between $\mathcal{E}(A, B)$ and $\mathrm{Ext}^1_R(A, B)$? Given an extension

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

we can get

$$\mathrm{Hom}_R(E, B) \longrightarrow \mathrm{Hom}_R(B, B) \xrightarrow{\partial} \mathrm{Ext}^1(A, B)$$

So if $\mathcal{E}$ is the class of our extension, we can send $\mathcal{E} \mapsto \partial(\mathrm{id}_B)$.

If $u \in \mathrm{Ext}^1_R(A, B)$, we have

$$0 \longrightarrow \ker \longrightarrow P \longrightarrow A \longrightarrow 0$$

where P is projective. This gives us an exact sequence

$$\mathrm{Hom}_R(K, B) \xrightarrow{\partial} \mathrm{Ext}^1_R(A, B) \longrightarrow 0 = \mathrm{Ext}^1_R(P, B)$$

So we can construct

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "t"] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where E is the push-out of P and B over K.

Theorem 1.2. *There exists a 1 to 1 corespondence between* $\mathrm{Ext}^i_R(A,B)$ *for* $i \geq 1$ *and equivalence classes of "Yoneda extensions" in* $\mathcal{E}_K(A,B)$*:*

$$0 \longrightarrow B \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \cdots \longrightarrow K_1 \longrightarrow A \longrightarrow 0$$

1.4 Group cohomology

Definition 1.4. Let G be a group, and let A be a $\mathbb{Z}[G]$-module. The G**-invariant group** of A is $A^G = \{a \in A : ga = a \,\forall g \in G\}$. The G**-coinvariant group** is $A_G = A/I_G A$, the largest quotient of A on which G acts trivially, where I_G is the **augmentation ideal**; $I_G = \ker(\mathcal{E})$, where $\mathcal{E} : \mathbb{Z}[G] \to \mathbb{Z}$ sendins $\sum_g a_g g \mapsto \sum_g a_g$.

Remark 1.2. The augmentation ideal is generated by $g-1$ for $g \in G$: $\sum a_g g \in I_G \iff \sum a_g = 0$. So $\sum a_g g - \sum a_g = \sum a_g(g-1)$.

Example 1.3. $\mathbb{Z}$ is a trivial G-module with $gn = n$. Then $\mathbb{Z}^G = \mathbb{Z}_G \cong \mathbb{Z}$.

Example 1.4. Consider $\mathbb{Z}[G]$. Then

$$\mathbb{Z}[G]^G = \begin{cases} (N_G) & G \text{ finite} \\ 0 & G \text{ infinite}, \end{cases} \qquad N_G = \sum_{g \in G} g,$$

$$\mathbb{Z}[G]_G \cong \mathbb{Z}[G]/I_G \cong \mathbb{Z}.$$

Definition 1.5. The i**-th cohomology group** of G with coefficients in A (a $\mathbb{Z}[G]$-module), $H^i(G,A)$, is the ith right derived functor of $M \to M^G$ on A. the i**-th homology group** of G with coefficients in A (a $\mathbb{Z}[G]$-module), $H^i(G,A)$, is the ith left derived functor of $M \to M_G$ on A.

We have the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \xrightarrow{\mathrm{ev}_1} A^G.$$

So $h_{\mathbb{Z}} \cong (\cdot)^A$. So we get

$$H^i(G,A) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Similarly, we have the isomorphism $A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$, so we get

$$H_i(G,A) \cong \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Definition 1.6. The **bar resolution** of $\mathbb{Z}$ in $\mathbb{Z}[G]$-Mod is the free resolution

$$\cdots \longrightarrow \mathbb{Z}[G^4] \xrightarrow{d_3} \mathbb{Z}[G^3] \xrightarrow{d_2} \mathbb{Z}[G^2] \xrightarrow{d_1} \mathbb{Z}[G, \text{"}\varepsilon\text{"}] \longrightarrow \mathbb{Z} \longrightarrow 0$$

denoted $C_{\cdot} \to \mathbb{Z}$ with $C_i = \mathbb{Z}[G^{i+1}]$ and $d_i : C_i \to C_{i-1}$. Here,

$$d_i((g_0, g_1, \ldots, g_i)) = \sum_{j=0}^{i} (-1)^j (g_0, \ldots, \widehat{g}_j, \ldots, g_j).$$

Then

$$0 \longrightarrow \underbrace{\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A)}_{\cong A} \xrightarrow{D^0} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^2], A) \xrightarrow{D^1} \cdots$$

computes $H^i(G, A)$.

This gives an **inhomogeneous cochain complex**

$$0 \longrightarrow C^0(G, A) \xrightarrow{d^0} C^0(G, A) \xrightarrow{d^1} \cdots$$

where $C^i(G, A) = \{f : G^i \to A\}$, and

$$d^i f(g_0, \ldots, g_i) = g_0 f(g_1, \ldots, g_i) + \sum_{j=1}^{i} (-1)^j f(g_1, \ldots, g - j - 1, g_i, \ldots, g_i) \\ + (-1)^{i+1} f(g_1, \ldots, g_{i-1}).$$

These complexes are isomorphic.

We have that

$$H^i(G, A) = \frac{Z^i(G, A)}{B^i(G, A)},$$

where $Z^i(G, A)$ are called **cocycles** and $B^i(G, A)$ are called **coboundaries**.